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INTERFERENCE BETWEEN THE TRANSVERSE AND LONGITUDINAL VIBRATIONS IN MUSICAL STRINGS†

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Two problems of the vibrations of strings are considered using the approach described previously in [1]: the vibrations of the string of a plucked musical instrument, drawn out at one of the points and at rest at the initial instant of time (Problem 1), and the vibrations of the string of a keyboard musical instrument, the points of which are given an initial velocity at the initial instant of time by a hammer of small width (Problem 2). It is established that forced longitudinal oscillations of the string occur at frequencies of the transverse vibrations, the condition for possible resonance of the longitudinal vibrations is derived, and the nature of the vibrations at the point where the string is fastened due to elasticity and the related shift in the frequency of transverse vibrations is established. © 2003 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM OF THE VIBRATIONS OF MUSICAL STRINGS

As is well known (for example, from [2, 3]), the basis of the investigation of the vibrations of musical strings is the equations of transverse vibrations, which establish, qualitatively correctly, the relation between the frequencies and the length of the string, its tension and density. Nevertheless, there are no reliable data that the spectrum of the vibrations predicted on the basis of these equations corresponds to the measured value. Due to the fact that the dynamic components of the tension in the string are not taken into account, the mechanism of the vibration of the sounding board – the fundamental generator of the waves – is not completely described. In this connection, based on the well-known equations [4].

$$\rho_0 x_{tt} = (T\cos\theta)_s, \quad \rho_0 y_{tt} = (T\sin\theta)_s$$
$$\cos\theta = \frac{1+x_s}{1+e}, \quad \sin\theta = \frac{y_s}{1+e}, \quad e = \sqrt{(1+x_s)^2+(y_s)^2}-1 \tag{1.1}$$

where s is the Lagrange coordinate of a particle, measured in the position when the string is not under tension and has a density ρ_0, $x(s, t)$ and $y(s, t)$ are the coordinates of the displacement vector, e is the deformation, $T = eE$ is the tension and E is Young's modulus, linearized equations were derived in [1] which enable the longitudinal and transverse vibrations of strings to be taken into account. The fact that longitudinal waves propagate (in addition to transverse waves) was pointed out previously in [4].

The displacement x can be conveniently represented in the form

$$x = \tilde{x} + x_0(s); \quad e = \tilde{e} + e_0, \quad x_0 = e_0 s$$

which denotes the reading of the value of $\tilde{x}$ (and also y) with respect to the fixed string, stretched to a deformation e_0 = const.

The unknown functions $\tilde{x}(s, t), y(s, t)$ can be sought in the form

$$y = \varepsilon^{1/2}(y_1 + \varepsilon y_2 + \ldots), \quad \tilde{x} = \varepsilon(x_1 + \varepsilon x_2 + \ldots), \quad \tilde{e} = \varepsilon z_1 + \varepsilon^2 z_2 + \ldots \tag{1.2}$$

where ε is the characteristic value of the additional deformation.

When Eqs (1.1) are expanded in the small parameter ε the following system of equations of the first approximation is obtained [1]

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$$y_{1tt} = b_0^2 y_{1ss}, \quad b_0^2 = \frac{T_0}{\rho_0(1+e_0)} \tag{1.3}$$

$$x_{1tt} = a_0^2\left(x_{1s} + \frac{1}{2(1+e_0)^2} y_{1s}^2\right)_s, \quad a_0^2 = \frac{E}{\rho_0} \tag{1.4}$$

Taking into account one or other of the initial conditions for $x_{1s}, y_{1s}, x_{1t}, y_{1t}$ and also the conditions at the points where the string is fastened, one can relate the nature of the action on the string to the transmission of vibrations into the sounding board.

Below we will solve two fundamental problems of the theory of the vibrations of musical strings, in which the traditional formulation for the longitudinal vibrations $y_1(s, t)$ is supplemented by the formulation of the problem for the vibrations $x_1(s, t)$.

Problem 1 (the string of a plucked instrument is fastened at points $s = 0$ and $s = 1$, and when $t = 0$ it is plucked at the point $s = c$ to a height h and then released)

$$y_1(0,t) = y_1(l,t) = y_{1t}(s,0) = x_1(0,t) = x_1(l,t) = x_{1t}(s,0) = 0$$

$$y_1(s,0) = \frac{h}{c}s, \quad x_1(s,0) = \frac{s}{l}(c - l + \xi c), \quad \text{if} \quad 0 \le s \le c$$

$$y_2(s,0) = \frac{h}{c-l}(s-l), \quad x_1(s,0) = \left(1 - \frac{s}{l}\right)\left(c - \frac{l-c}{\xi}\right), \quad \text{if} \quad c \le s \le l \tag{1.5}$$

$$\xi = \frac{\sqrt{h^2 + (l-c)^2}}{\sqrt{h^2 + c^2}}$$

Problem 2 (the string of a keyboard instrument is fastened at the points $s = 0$ and $s = l$, and at $t = 0$ a hammer of width 2δ gives the particles at rest a velocity $V_0 = \text{const}$)

$$y_1(0,t) = y_1(l,t) = y_1(s,0) = x_1(0,t) = x_1(l,t) = x_{1t}(s,0) = 0$$

$$y_{1t}(s,0) = \begin{cases} V_0, & \text{if} \quad c - \delta \le s \le c + \delta \\ 0, & \text{if} \quad s \notin [c - \delta, c + \delta] \end{cases} \tag{1.6}$$

2. THE PROBLEM OF THE INITIAL PHASE OF THE MOTION OF A STRING IN THE NON-LINEAR AND LINEAR FORMULATIONS

In Problem 1, to give an initial triangular form to the string it is necessary to apply a force F_0 to the point C (Fig. 1). This force causes, in sections AC and BC, an additional deformation

$$e_0^{(1)} = (AC + BC - AB)/AB$$

(This quantity $e_0^{(1)}$ corresponds to the case of slippage of the string at the point C.)

Suppose the string is released at $t = 0$; then, before faster longitudinal waves arrive at the ends A and B, the solution corresponds to the case of the propagation of waves in a string that is unconstrained on both sides (the parts which are inclined at angles of θ_1 and θ_2 to the X axis). Due to the fact that in this problem there is no characteristic length and the initial deformation is not constant, dimensional analysis gives

$$y = tf_1(s/(a_0 t)), \quad x = tf_2(s/(a_0 t))$$

In this case Eqs (1.3) and (1.4), as was shown previously [4, 5], have the simplest solutions $f_1' = \text{const}$ and $f_2' = \text{const}$, which denote that the components of the velocities and the deformations are constant. Along the parts of the string of the initial form, longitudinal waves L_1 and L_2 propagate from the point C, and after them transverse waves S_1 and S_2. At the point C deflection of the string is impossible

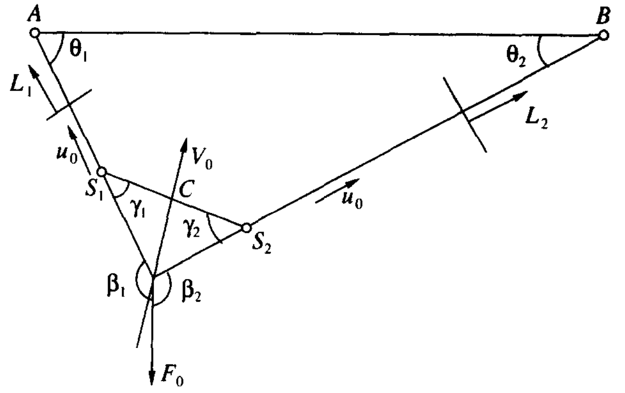

Fig. 1

(otherwise an element in the neighbourhood of the point C moves with an acceleration, which contradicts the solutions for the velocity). Hence, along the section S_1S_2 the deformations (tensions) and the velocities of all the particles are the same and constant. As was shown in [4, 5], the deformations do not suffer discontinuities for transverse waves, and hence everywhere in the region $L_1S_1CS_2L_2$ the deformation $e_1^{(1)}$ (in addition to e_0) is constant. Due to the difference in the values of $e_0^{(1)}$ and $e_1^{(1)}$ along the sections L_1S_1 and L_2S_2 there is longitudinal motion of the particles of the string with velocity $u_0 = a_0(e_0^{(1)}, e_1^{(1)})$ towards the points A and B (as will be seen later $e_0^{(1)} > e_1^{(1)}$).

The use of the law of the change in momentum in the direction AS_1 and perpendicular to it gives [4, 5]

$$\rho_0(b_1 - u_0)(V_0\cos\beta_1 + u_0) = T(e)(\cos\gamma_1 - 1)(1+e), \quad e = e_0 + e_1^{(1)} \tag{2.1}$$

$$\rho_0(b_1 - u_0)V_0\sin\beta_1 = T(e)\sin\gamma_1(1+e) \tag{2.2}$$

Here b_1 is the velocity of the wave S_1 and V_0 is the modulus of the velocity along section S_1S_2. The fact that the angle of deflection of the string $L_1S_1S_2$ remains unchanged with time leads to the relation

$$b_1\sin\gamma_1 = V_0\sin(\beta_1 - \gamma_1) \tag{2.3}$$

We can obtain the following relations [4, 5] from Eqs (2.1)–(2.3)

$$\rho_0(b_1 - u_0)^2 = Ee(1+e) = T(e)\sin\gamma_1 \tag{2.4}$$

$$\begin{aligned} b_1 &= b_2 = b = a_0\sqrt{(e_0 + e_1^{(1)})(1 + e_0 + e_1^{(1)})} + a_0(e_0^{(1)} - e_1^{(1)}) \\ V_0\sin\beta_1 &= (b - u_0)\sin\gamma_1 \end{aligned} \tag{2.5}$$

From Eqs (2.1) and (2.2) and similar equations for the wave S_2, taking the relations $\theta_1 - \gamma_1 = \gamma_2 - \theta_2$, $\beta_1 + \beta_2 = \pi + \gamma_1 + \gamma_2$ into account, we obtain V_0, $e_0^{(1)}$, γ_1, β_1.

Taking into account the fact that θ_0 is small, we have

$$\begin{aligned} &b\sin\theta_0 = V_0, \quad V_0\cos\theta_0 = (b - u_0)\sin\theta_0 \\ &b\cos\theta_0 = b - u_0, \quad \frac{\hat{\varepsilon}}{1-\cos\theta_0} = \sqrt{(e_0 + e_1^{(1)})(1 + e_0 + e_1^{(1)})} + \hat{\varepsilon}, \quad \hat{\varepsilon} = e_0^{(1)} - e_1^{(1)} \end{aligned} \tag{2.6}$$

$$\begin{aligned} &1 - \cos\theta_0 \approx \frac{\theta_0^2}{2}, \quad \hat{\varepsilon} \approx \frac{\theta_0^2}{2}\sqrt{e_0(1+e_0)}, \quad \bar{b}_0 = \frac{b_0}{a_0} = \sqrt{e_0(1+e_0)} \\ &\bar{V}_0 = \bar{b}_0\theta_0 \approx \frac{\sqrt{2\hat{\varepsilon}}}{\sqrt{\bar{b}_0}}\bar{b}_0 = \sqrt{2\hat{\varepsilon}\bar{b}_0}, \quad \bar{V}_0 \approx \sqrt{\hat{\varepsilon}} \end{aligned} \tag{2.7}$$

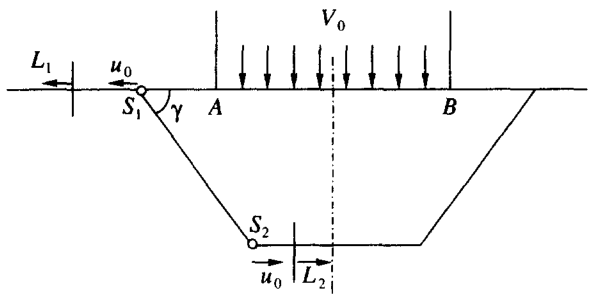

Fig. 2

In Problem 2, at the instant $t = 0$, two longitudinal waves (L_1 and L_2) and two transverse waves (S_1 and S_2) begin to propagate from points A and B (Fig. 2). Until one of the waves L_1 and L_2 reaches the point where the string is fastened or the middle AB, the solution corresponds to the problem of the propagation of waves in an unlimited string, to all points of which, beginning from A and further, a constant velocity V_0 is applied from the right at the initial instant. As in Problem 1, it can be shown that in the region $L_1S_1S_2L_2$ the stretching deformation (additional to e_0) is constant, and along the parts L_1S_1 and L_2S_2 longitudinal motion occurs with a velocity u_0 to the deflection points.

The components of the velocity V_x and V_y along the section S_1S_2 are constant. The use of the laws of the change in momentum along the direction OX when the particles pass through the waves S_1 and S_2, respectively, leads to the following relations, similar to (2.1)

$$\rho_0(b+u_0)(\pm V_x - u_0) = T\cos\gamma(\cos\gamma - 1)(1+e) \quad (e = e_0 + e_1^{(1)}) \tag{2.8}$$

whence $V_x = 0$.

Similarly, along the direction OY, like (2.2) we have

$$\rho_0(b+u_0)V_y = T\sin\gamma(1+e), \; \rho_0(b+u_0)(V_y - V_0) = -T\sin\gamma(1+e); \; V_y = V_0/2 \tag{2.9}$$

From the remaining equations (2.8) and (2.9), which we write in the form

$$\rho_0(b+u_0)u_0 = T(1-\cos\gamma)(1+e), \quad \rho_0(b+u_0)V_0/2 = T\sin\gamma(1+e) \tag{2.10}$$

and from the relations b tg$\gamma = V_0/2$, $u_0 = a_0e^{(1)}$ when $\bar{V}_0 = V_0/a_0 \ll 1$ we obtain

$$\bar{b} = \frac{b}{a_0} \approx \sqrt{e_0}, \gamma \approx \frac{1}{2}\frac{\bar{V}_0}{\sqrt{e_0}}, e_1^{(1)} \approx \frac{\bar{V}_0^2}{8\sqrt{e_0(1+e_0)}}, \bar{V}_0 \approx 2\sqrt{2}[e_0(1+e_0)]^{1/4}\sqrt{e^{(1)}} \tag{2.11}$$

We will solve Problem 2 within the framework of Eqs (1.3) and (1.4). Taking condition (2.4) into account, we convert relations (2.9) as follows:

$$V_y = (b_0+u_0)\sin\gamma, \quad V_y - V_0 = -(b_0+u_0)\sin\gamma \tag{2.12}$$

Denoting the trajectory of the transverse wave by $s = s^*(t)$, we have [4]

$$b = \frac{d}{dt}\{s^*(t) + x[s^*(t), t]\} = \frac{ds^*(t)}{dt}(1+x_s) - u$$

$$(b+u) = (1+e)\frac{ds^*}{dt}; \quad \frac{ds^*}{dt} = \pm\sqrt{\frac{T}{\rho_0(1+e)}}$$

In this case relations (2.12) take the following form

$$V_y = y_t = -\frac{ds^*}{dt}y_s, \quad V_y - V_0 = \frac{ds^*}{dt}y_s \tag{2.13}$$

which are identical for the transverse components $y_1(s, t)$ with the relations on the characteristics $ds^*/dt = b_0$ for Eq. (1.3). Hence, relations (2.13) do not introduce any additional information into the formulation of problem 2 for Eq. (1.3). Its solution as $t \to 0$ (when the wave pattern is identical with the pattern which occurs in the case when a constant velocity V_0 is given to all points $s < 0$ to the left of A when $t = 0$) has the form

$$\begin{aligned} &y_{1s} = 0, \quad \sqrt{e_1^{(1)}}y_{1t} = V_0 \quad \text{when} \quad -\infty < s < -b_0 t \\ &y_{1t} = y_{1s} = 0 \quad \text{when} \quad s^* \geq b_0 t \\ &\sqrt{e_1^{(1)}}y_{1t} = \frac{V_0}{2}, \quad \sqrt{e_1^{(1)}}y_{1s} = -\frac{V_0}{2b_0} \quad \text{when} \quad -b_0 t \leq s^* \leq b_0 t \end{aligned} \tag{2.14}$$

The value $V_y = V_0/2$ in the region of the deflection, obtained from (2.14), is identical with its exact value.

To find $x_1(s, t)$ we convert relation (1.8) in the same way as was done with relations (2.13). We obtain

$$\begin{aligned} V_x - u_0 &= (b + u_0)(\cos\gamma - 1) = \frac{ds^*}{dt}(1 + e)(\cos\gamma - 1) = -\frac{ds^*}{dt}(e_1^{(1)} - \bar{x}_s) \\ -V_x - u_0 &= (b + u_0)(\cos\gamma - 1) = \frac{ds^*}{dt}(1 + e)(\cos\gamma - 1) = -\frac{ds^*}{dt}(e_1^{(1)} - \bar{x}_s) \end{aligned} \tag{2.15}$$

Taking into account that, for the first approximation

$$e_1^{(1)} = x_{1s} + \frac{1}{2(1 + e_0)}y_{1s}^2$$

relations (2.15) can be written in the form

$$x_{1t} - a_0 = -\frac{b_0}{2(1 + e_0)}y_{1s}^2, \quad -x_{1t} - a_0 = -\frac{b_0}{2(1 + e_0)}y_{1s}^2 \tag{2.16}$$

whence $x_{1s} = 0$ behind the transverse waves. In this case

$$\frac{2(1 + e_0)^{3/2}}{\sqrt{e_0}} = y_{1s}^2, \quad \gamma \sim \frac{\sqrt{\varepsilon_1^{(1)}}y_{1s}}{1 + e_0} = \frac{\sqrt{2}\sqrt{\varepsilon_1^{(1)}}}{[e_0(1 + e_0)]^{1/4}}$$

Since $\operatorname{tg}\gamma = V_0(2b_0)^{-1}$, we have

$$\bar{V}_0 \approx 2\bar{b}\gamma = 2\sqrt{2}[\varepsilon_1^{(1)}]^{1/4}[e_0(1 + e_0)]^{1/4}$$

which is identical with relations (2.11).

The results obtained are based on the fact that the values of the components of the velocities and deformations, which arise from the self-similar problem, are constant.

It we do not use this information, we have

$$\begin{aligned} &x_1 = f_1\left(t - \frac{s}{a_0}\right) \quad \text{when} \quad b_0 t \leq s \leq a_0 t, \quad x_1 = f_2\left(t + \frac{s}{a_0}\right) \quad \text{when} \quad -a_0 t \leq s \leq -b_0 t \\ &x_1 = f_3\left(t - \frac{s}{a_0}\right) + f_4\left(t + \frac{s}{a_0}\right) \quad \text{when} \quad -b_0 t \leq s \leq b_0 t \end{aligned}$$

The functions f_1', f_2', f_3', f_4' are found from the two relations of (2.16) and two relations that express the continuity of the deformation on the transverse waves, which is related to the previous result.

We will derive the solution of Problem 1. The conditions on the characteristics

$$y_{1t} = -b_0\left[y_{1s} + (1 + e_0)\frac{\theta}{\bar{\varepsilon}^{1/2}}\right], \quad \frac{\partial s}{\partial t} = -b_0; \quad y_{1y} = b_0\left[y_{1s} + (1 + e_0)\frac{\theta_1}{\bar{\varepsilon}^{1/2}}\right], \quad \frac{\partial s}{\partial t} = b_0$$

define the velocity

$$y_{1t} = -b_0(1+e_0)(4\varepsilon)^{-1/2}(\theta_1+\theta_2)$$

and the quantity y_{1s} on S_1S_2. When $\theta_1 = \theta_2 = \theta_0$ we have

$$y_{1t} = -b_0(1+e_0)\varepsilon^{-1/2}\theta_0, \quad y_{1s} = 0$$

and, from the solution of Eq. (1.4), we obtain $x_{1t} = 0$ in the region S_1S_2 and $x_{1t} = \text{const}$ in the regions L_1S_1 and L_2S_2.

It is necessary to take into account the effect of the stiffness of the string, but the regions where this has an effect will be of the order of several diameters of the string and, in our opinion, will have no appreciable influence on the vibrations.

3. THE SPECTRA OF THE TRANSVERSE AND LONGITUDINAL VIBRATIONS OF MUSICAL STRINGS

It is well known [3] that the solution $y_1(s, t)$ of problem 1 has the form

$$y_1(s,t) = \sum_{n=0}^{\infty} A_n \sin\frac{\pi ns}{l}, \quad A_n = \frac{2hl^2}{\pi^2 n^2(l-c)c}\sin\frac{\pi ns}{l}\cos\omega_n t, \quad \omega_n = \frac{\pi nb_0}{l} \tag{3.1}$$

Calculations of

$$y_{1s} = \sum_{n=1}^{\infty} B_n \cos\frac{\pi ns}{l}, \quad B_n = A_n\frac{\pi n}{l}$$

$$y_{1s}^2 = \sum_{n=1}^{\infty}\frac{B_n^2}{2}\left(1+\cos\frac{2\pi ns}{l}\right) + \sum_{i\neq j}\frac{B_iB_j}{2}\left[\cos\frac{\pi s(i+j)}{l} + \cos\frac{\pi s(i-j)}{l}\right]$$

$$(y_{1s}^2)_s = -\sum_{n=1}^{\infty}\frac{B_n^2\pi n}{l}\sin\frac{2\pi ns}{l} - \sum_{i\neq j}\frac{B_iB_j\pi(i+j)}{2l}\sin\frac{\pi s(i+j)}{l} -$$

$$-\sum_{i\neq j}\frac{B_iB_j\pi(i-j)}{2l}\sin\frac{\pi s(i-j)}{l}$$

(here and henceforth $i \geqslant 1, j \geqslant 1$) show that the load per unit length, which occurs in the right-hand side or Eq. (1.4), is the superposition of component of the transverse harmonics.

Taking into account the fact that

$$B_i = \frac{\pi i}{l}\frac{2hl^2}{\pi^2 i^2(l-c)c}\sin\frac{\pi ic}{l}\cos\omega_i t; \quad \omega_{i+j} = \omega_i + \omega_j, \quad \omega_{i-j} = \omega_i - \omega_j$$

the coefficients of the even and odd harmonics can be represented in the form

$$P_{2m}(t) = -\frac{D}{m}\sin^2\left(\frac{\pi mc}{l}\right)(\cos 2\omega_m t + 1) - \sum_{i-j=2m}\frac{2mD}{ij}L_{ij}E_{ij} - \sum_{i+j=2m}\frac{2mD}{ij}L_{ij}E_{ij} \tag{3.2}$$

$$P_{2m-1}(t) = -\sum_{i-j=2m-1} D\frac{(2m-1)}{ij}L_{ij}E_{ij} - \sum_{\substack{i+j=2m-1\\ i<j}} D\frac{(2m-1)}{ij}L_{ij}E_{ij}$$

$$D = \frac{2h^2 l}{\pi c^2(l-c)^2}, \quad L_{ij} = \sin\frac{\pi ic}{l}\sin\frac{\pi jc}{l}, \quad E_{ij} = (\cos\omega_{i+j}t + \cos\omega_{i-j}t) \tag{3.3}$$

The solution of homogeneous equation (1.4) with conditions (1.5) has the form

$$\varphi_n = \frac{2}{\pi^2 n^2}\sin\frac{\pi nc}{l}W + \frac{2}{\pi n}\cos\frac{\pi nc}{l}\left\{c - \frac{(l-c)}{\xi} - \frac{c}{l}W\right\}$$
$$u^{(I)} = \sum_{n=1}^{\infty}\varphi_n\cos(\omega_n^* t)\sin\frac{\pi nS}{l}, \quad \omega_n^* = \frac{\pi n a_0}{l}, \quad W = 2c - l + \xi c - \frac{(l-c)}{\xi} \tag{3.4}$$

where ω_n^* is the frequency of longitudinal vibrations.

For even harmonics ($n = 2m$) the solution of Eq. (1.4)

$$x_{1tt} - a_0^2 x_{1ss} = \frac{a_0^2}{2(1+e_0)^2}P_{2m}(t)\sin\frac{2m\pi S}{l}$$

where $P_{2m}(t)$ is an expression of the form (3.2), has the form

$$u_{2m}^{(II)}(s,t) = F_{2m}(t)\sin\frac{2m\pi s}{l}$$

The function F_{2m} is found from the equation

$$\frac{d^2F_{2m}}{dt^2} + a_0^2\frac{4m^2\pi^2}{l^2}F_{2m} = KP_{2m}(t), \quad K = \frac{a_0^2}{2(1+e_0)^2}$$

Then

$$F_{2m}(t) = \frac{1}{\omega_{2m}^*}\int_0^t KP_{2m}(\tau)\sin(\omega_{2m}^*(t-\tau))d\tau$$

$$F_{2m} = -\frac{1}{\omega_{2m}^*}K\left[\frac{D}{m}\sin^2\frac{2\pi mc}{l}\left\{\frac{\omega_{2m}^*}{\omega_{2m}^{*2} - 4\omega_m^2}\cos 2\omega_m t + \frac{1}{\omega_{2m}^*} + \right.\right.$$
$$\left.\left. + \frac{4\omega_m^2 - 2\omega_{2m}^{*2}}{\omega_{2m}^*(\omega_{2m}^{*2} - 4\omega_m^2)}\cos\omega_{2m}^* t\right\} + \sum_{i-j=2m}\frac{2Dm}{ij}L_{ij}A_{i,j}^{2m} + \sum_{\substack{i-j=2m\\ i<j}}\frac{2Dm}{ij}L_{ij}A_{i,j}^{2m}\right] \tag{3.5}$$

where

$$A_{i,j}^m = \frac{\omega_m^*}{\omega_m^{*2} - \omega_{i+j}^2}\cos\omega_{i+j}t + \frac{\omega_m^*}{\omega_m^{*2} - \omega_{i-j}^2}\cos\omega_{i-j}t + \frac{\omega_m^*(\omega_{i-j}^2 + \omega_{i+j}^2 - 2\omega_m^{*2})}{(\omega_{2m}^{*2} - \omega_{i+j}^2)(\omega_m^{*2} - \omega_{i-j}^2)}\cos\omega_m^* t$$
$$x_{12m} = u_{2m}^{(I)} + u_{2m}^{(II)} \tag{3.6}$$

The method of obtaining F_{2m-1} for the odd harmonics is similar. We obtain

$$F_{2m-1} = -\frac{1}{\omega_{2m-1}^*}K\left[\sum_{i-j=2m-1}\frac{D(2m-1)}{ij}L_{ij}A_{i,j}^{2m-1} + \sum_{\substack{i-j=2m-1\\ i<j}}\frac{D(2m-1)}{ij}L_{ij}A_{i,j}^{2m-1}\right] \tag{3.7}$$

The general solution for $x_1(s, t)$ has the form

$$x_1 = \sum_{m=1}^{\infty}(x_{12m} + x_{12m-1}), \quad x_{12m} = \varphi_{2m}\cos\omega_{2m}^* t\sin\frac{2\pi ms}{l} + F_{2m}\sin\frac{2\pi ms}{l}$$
$$x_{12m-1} = \varphi_{2m-1}\cos\omega_{2m-1}^* t\sin\frac{(2m-1)\pi s}{l} + F_{2m-1}\sin\frac{(2m-1)\pi s}{l} \tag{3.8}$$

Expressing i in terms of j (or j in terms of i) with $i \geqslant 1, j \geqslant 1$, in relations (3.5) and (3.7) the displacement along x can be represented in the form

$$x_{12m} = \left\{\left(\frac{1}{2\pi^2 m^2}\sin\frac{2\pi mc}{l}W + \frac{1}{\pi m}\cos\frac{2\pi mc}{l}\left(c - \frac{l-c}{\xi} - \frac{c}{l}W\right)\right)\cos\omega_{2m}^* t - \right.$$
$$- \frac{1}{\omega_{2m}^*}KD\left[\frac{1}{m}\sin^2\frac{2\pi mc}{l}\left(\frac{\omega_{2m}^*}{\omega_{2m}^{*2} - 4\omega_m^2}\cos 2\omega_m t + \frac{1}{\omega_{2m}^*} + \frac{4\omega_m^4 - 2\omega_{2m}^{*2}}{\omega_{2m}^*(\omega_{2m}^{*2} - 4\omega_m^2)}\cos\omega_{2m}^* t\right) + \right. \tag{3.9}$$
$$\left.\left. + \sum_{j=1}^{\infty}\frac{2m}{(2m+j)j}L_{2m+j,j}A_{2m+j,j}^{2m} + \sum_{i=1}^{m-1}\frac{2m}{i(2m-i)}L_{i,2m-i}A_{i,2m-i}^{2m} + \right]\right\}\sin\frac{2\pi ms}{l}$$

$$x_{12m-1} = \left\{\left(\frac{2}{\pi^2(2m-1)^2}\sin\frac{\pi(2m-1)c}{l}W + \right.\right.$$
$$\left. + \frac{2}{\pi(2m-1)}\cos\frac{\pi(2m-1)c}{l}\left(c - \frac{l-c}{\xi} + \frac{c}{l}W\right)\right)\cos\omega_{2m-1}^* t -$$
$$- \frac{1}{\omega_{2m-1}^*}KD\left[\sum_{j=1}^{\infty}\frac{2m}{(2m-1+j)j}L_{2m-1+j,j}A_{2m-1+j,j}^{2m-1} + \right. \tag{3.10}$$
$$\left.\left. + \sum_{i=1}^{m-1}\frac{2m}{(2m-i-1)i}L_{2m-1-i,i}A_{i,2m-i-1}^{2m-1}\right]\right\}\sin\frac{\pi(2m-1)s}{l}$$

The determination of the characteristics of the vibrations of the string in Problem 2 is similar, and hence below we will only give the main stages of the solution.

In Problem 2, as is well known [2]

$$y_1(s,t) = \frac{4V_0}{b_0\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi nc}{l}\sin\frac{\pi n\delta}{l}\sin\frac{\pi ns}{l}\sin\frac{\pi nb_0 t}{l}$$

The solution of the problem of the longitudinal vibrations has the form (3.8), where

$$\varphi_{2m} = \varphi_{2m-1} = 0$$

$$F_{2m} = -\frac{8V_0^2}{\omega_{2m}^* l^3 b_0^3 \pi m}\sin^2\frac{2\pi mc}{l}\sin^2\frac{2\pi m\delta}{l}K\left[\frac{\omega_{2m}^*}{\omega_{2m}^{*2} - 4\omega_m^2}\cos 2\omega_m t + \right.$$
$$\left. + \frac{4\omega_m^2 - 2\omega_{2m}^{*2}}{\omega_{2m}^*(\omega_{2m}^{*2} - 4\omega_m^2)}\cos\omega_{2m}^* t + \frac{1}{\omega_{2m}^*}\right] -$$
$$- \frac{K}{\omega_{2m}^*}\sin\frac{\pi ic}{l} + \left[\sum_{i-j=2m}\frac{32mV_0^2}{l^3 b_0^2\pi ij}L_{ij}J_{ij}A_{i,j}^{2m} + \sum_{i+j=2m}\frac{32mV_0^2}{l^3 b_0^2\pi ij}L_{ij}J_{ij}A_{i,j}^{2m}\right]$$

$$F_{2m-1} = -\frac{K}{\omega_{2m-1}^*}\frac{16V_0^2}{\pi l^3 b_0^2}\left[\sum_{i-j=2m-1}\frac{(2m-1)}{ij}L_{ij}J_{ij}A_{i,j}^{2m-1} + \sum_{i+j=2m-1}\frac{(2m-1)}{ij}L_{ij}J_{ij}A_{i,j}^{2m-1}\right]$$

$$J_{ij} = \sin\frac{\pi i\delta}{l}\sin\frac{\pi j\delta}{l}$$

4. THE EFFECT OF ELASTIC CLAMPING

We will consider non-zero boundary conditions at the clamping points. Suppose the clamping is rigid when $s = l$: $y = \bar{x} = 0$, and is elastic when $s = 0$. We have for the displacement vector

$$\mathbf{l} = k\mathbf{T} \tag{4.1}$$

Condition (4.1) for y and $\bar{x}$ takes the form

$$y = kE\hat{e}\sin\theta = kE(e_0 + x_s)y_s \tag{4.2}$$

$$\bar{x} = kE\hat{e}\cos\theta, \quad x_1 = k\left(x_s + \frac{1}{2(1+e_0)}y_s^2\right) \tag{4.3}$$

The natural assumption $k \ll 1$ enables us to seek solutions in the form

$$\begin{aligned} x_1(s,t) &= x_{10}(s,t) + kx_{11}(s,t) + k^2x_{12}(s,t) \\ y_1(s,t) &= y_{10}(s,t) + ky_{11}(s,t) + k^2y_{12}(s,t) \end{aligned} \tag{4.4}$$

The solutions for $y_{10}(s, t)$ and $x_{10}(s, t)$ are identical with solutions (3.1) and (3.8) respectively. The problems for determining $y_{11}(s, t)$ and $x_{11}(s, t)$ are as follows:

$$y_{11tt}(s,t) = b_0^2 y_{11ss}(s,t); \quad x_{11tt}(s,t) = a_0^2 x_{11ss}(s,t)$$

$$y_{11}(s,0) = y_{11t}(s,0) = y_{11}(0,t) = x_{11}(s,0) = x_{11t}(s,0) = x_{11}(0,t) = 0$$

$$y_{11}(l,t) = Ee_0 y_{10s}(l,t); \quad x_{11}(l,t) = x_{10s}(l,t) + \frac{1}{2(1+e_0)}y_{10s}^2(l,t)$$

The solution obtained by the method of separation of variables has the form

$$y_{11}(s,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}[y_{11n}^0(t) + ky_{11n}^1(t)]\sin\frac{\pi ns}{l} \tag{4.5}$$

where

$$y_{11n}^0(t) = \sum_{p \neq n} \frac{(-1)^{p+1}2h\sin\frac{\pi pc}{l}}{(\omega_p^2 - \omega_n^2)(l-c)c}(\cos\omega_p t - \cos\omega_n t) \tag{4.6}$$

$$y_{11n}^1(t) = \frac{(-1)^n h\sin\frac{\pi nc}{l}}{\omega_n(l-c)c}t\sin\omega_n t \tag{4.7}$$

The secular terms can be eliminated by renormalization [6], changing from ω_n to $\omega_n = \omega_n(1 + \mu)$. Expansion of solution (4.6) in the small parameter μ gives

$$y_{11}^0(s,t) = \sum_{p \neq n} \frac{(-1)^{p+1}2h\sin\frac{\pi pc}{l}}{(\omega_p^2 - \omega_n^2)(l-c)c}(\cos\omega_p' t - \cos\omega_n' t + \mu\omega_n' t\sin\omega_n' t) \tag{4.8}$$

This leads to the following result

$$y_{11}(s,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\sum_{p \neq n} \frac{(-1)^{p+1}2h\sin\frac{\pi pc}{l}}{(\omega_p^2 - \omega_n^2)(l-c)c}(\cos\omega_p' t - \cos\omega_n' t)\right]\sin\frac{\pi ns}{l}$$

where

$$\omega_i' = \omega_i(1+\mu), \quad \mu = -k(-1)^{i+1}\sin\frac{\pi nc}{l}\left[\omega_i \sum_{p \neq n} \frac{2(-1)^{p+1}}{\omega_p^2 - \omega_i^2}\sin\frac{\pi pc}{l}\right]^{-1}$$

since the secular terms from relation (4.8) are here cancelled out with the analogous terms from (4.7)

The solution for $x_{11}(s, t)$ is not given here in view of its complexity, apart from the expression for the frequency shift; $\Delta\omega^*_{i+j} \sim \frac{1}{2}(\omega_i + \omega_j)^{-2}$.

5. ANALYSIS OF THE SOLUTIONS

The results in Sections 3 and 4 enable us to draw the following conclusions.

1. The forced longitudinal vibrations contain frequencies of transverse vibrations.

2. Discontinuities on the transverse waves of the components of the longitudinal velocities and deformations are the reason for the occurrence of forced longitudinal vibrations at frequencies of the transverse vibrations. The solution obtained in the form of Fourier series for short times agrees with the solution in Section 2.

3. The spectrum of the vibrations also contains higher frequencies of the longitudinal vibrations of the string. For example, for the physical-mechanical parameters of a metal string, the note D of the first octave of a guitar ($E = 2 \times 10^{11}$ Pa, $\rho = 7850$ kg/m^3, $T = 82$ N, $l = 0.65$ m, the cross-section area of the string 7.069×10^{-8} m^2, $a_0 = 5048$ m/s and $b_0 = 383.31$ m/s) [7], $\omega^*_1 = 3883$ Hz, which exceeds $\omega_1 = 294$ Hz by practically a factor of 13. The subcontraoctave, the contraoctave, the major octave, and the lower, first, second, third, fourth and fifth octaves, as is well known [7], have the following frequencies (in Hz): 16.35–30.87, 32.4–61.74, 65.41–123.47, 130.81–246.94, 261.63–493.88, 523.25–987.77, 1046.5–1975.53, 2093–3951.07 and 4186.01–7902.13. Hence ω^*_1 lies in the fourth octave and must be taken into account (as also the next three frequencies) in the overall spectrum of the vibrations.

4. There is a shift in the natural frequencies of the vibrations due to the elasticity of the clamping.

5. Vibrations of the sounding board occur at frequencies close to the frequencies of the longitudinal and transverse vibrations.

6. In the expression for $x_1(s, t)$ (3.8) there is a component $A^m_{i,j}$ (3.6), from which the resonance condition can be determined

$$m = \frac{\sqrt{e_0}j}{\sqrt{1+e_0} - \sqrt{e_0}}, \quad m = \frac{\sqrt{e_0}i}{\sqrt{1+e_0} + \sqrt{e_0}}, \quad m = \frac{\sqrt{e_0}(2j-1)}{2(\sqrt{1+e_0} - \sqrt{e_0})}$$

$$m = \frac{\sqrt{e_0}(2i+1)}{2(\sqrt{1+e_0} - \sqrt{e_0})}, \quad i \in 1, 2, \dots, m-1, \quad j \in 1, 2, \dots$$

For example, when $e_0 = 1/197$ we have $\omega^*_1 = \omega_{13}$.

7. If the longitudinal vibrations are taken into account (including the forced vibrations), a new procedure for calculating the vibrations of musical instruments is required.

8. Experimental research in this area is desirable.

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